DECEMBER 15, 1941 PHYSICAL REVIEW VOLUME 60

Theory of the Elasticity of Polycrystals with Viscous Grain Boundaries*

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(Received October 16, 1941)

Previous experiments have shown that in annealed polycrystalline specimens, the grain boundaries are much more susceptible to plastic shear than are the interiors of the grains. This paper investigates the elastic properties of a specimen in which the grain boundaries are incapable of supporting shearing stresses. In such a specimen Young's modulus is found to be from 50 percent to 76 percent of its value when no slip at grain boundaries occurs, depending upon Poisson's ratio. This reduction of Young's modulus by slipping at grain boundaries should be observable by comparing values measured statically at elevated temperatures with those measured dynamically.

At high temperatures rupture occurs along grain boundaries, at low temperatures across the interior of the grains.[1] On the other hand, the shearing resistance of grain boundaries is apparently less at all temperatures than the shearing resistance of the individual grains themselves. Thus, according to Kanter,[2] a microsection of a lead specimen which has suffered some creep "presents the appearance of grains swimming in their own boundaries." The concept that slip along grain boundaries occurs with relative ease has been used by Schumacher[3] to explain the discontinuous creep character of large grained specimens at low stress levels. Finally, the variation of internal friction with temperature and with grain size[4] indicates that at low stress levels the individual grains behave elastically, but that they slip comparatively readily over one another.

A polycrystalline specimen under a constant load cannot of course continue to creep indefinitely merely by slipping at grain boundaries. As Schumacher[3] has pointed out, the grains form a self-locking system. Rather, the relaxation of shearing stresses across the grain boundaries gives rise to a "relaxed" Young's modulus, E_R, which is smaller than the Young's modulus when the stresses are unrelaxed, E. The measured modulus will lie between E_R and E, the precise value depending upon the time taken in its measurement, i.e., depending upon how much stress relaxation is allowed. Thus in measurements at elevated temperatures, E_R will be obtained by quasi-static methods, E by high frequency dynamical methods. In cyclic vibrations this stress relaxation will result in a dissipation of energy. The internal friction (ratio of imaginary to real part of modulus) resulting therefrom will have a maximum value of the order of magnitude of $(E-E_R)/(E+E_R)$. This maximum value will occur in the frequency range for which the period of vibration is comparable to the time of relaxation of the shearing stresses.

The purpose of the present paper is to compare the Young's modulus of a polycrystalline specimen in the case where the grain boundaries are slippery (E_R), with the case where no slip occurs across the boundaries (E). In §1 the essential features of the problem are analyzed. By neglecting all irrelevant complicating factors, it is found possible to reduce the problem to one that is soluble by the standard methods of elasticity

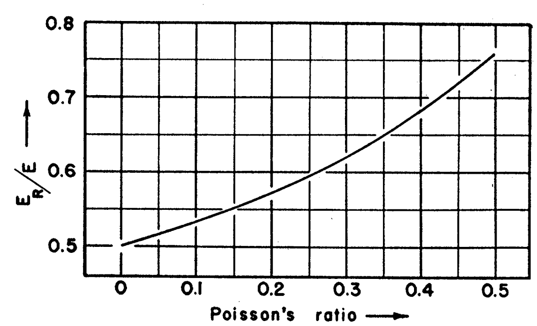

FIG. 1. Dependence of relaxed Young's modulus upon Poisson's ratio.

* This work was financed in part by the Engineering Foundation and was sponsored by the American Institute of Mining and Metallurgical Engineers.

[1] Z. Jeffries and R. Archer, *The Science of Metals* (McGraw-Hill, 1924), pp. 166–167.

[2] J. J. Kanter, Trans. A. I. M. E. **131**, 396 (1938).

[3] E. E. Schumacher, Trans. A. I. M. E. **143**, 176 (1941).

[4] A. Barnes and C. Zener, Phys. Rev. **58**, 87 (1940); C. Zener, D. Van Winkle, H. Nielsen, in press, Trans. A. I. M. E.

theory. The solution of this simplified problem is given in §2. The ratio E_R/E is obtained as a function of Poisson's ratio σ, Eq. (8). A graph of this ratio is given in Fig. 1.

§1. Formulation of Problem

The grain structure of a recrystallized polycrystalline metal has been compared to the cell structure of foam.[5] In each case the average unit is a pentagonal dodecahedron. Examination shows that slipping at the grain boundaries can occur only on the faces. The geometrical arrangement blocks slipping at the corners. This may be readily visualized by a two-dimensional structure, as in Fig. 2.

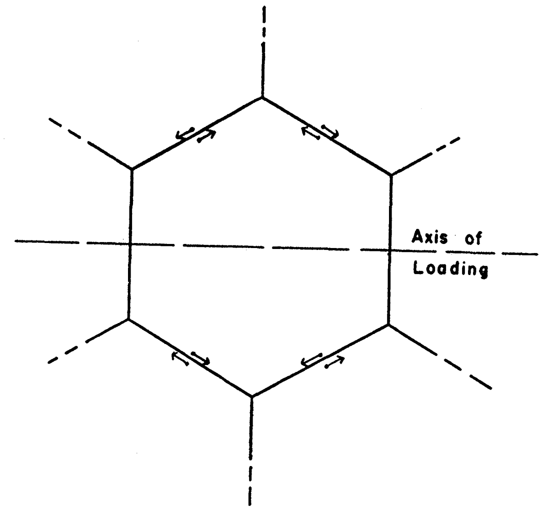

Fig. 2. Illustration of slipping at grain boundaries in a two-dimensional lattice. The arrows give the directions of slip.

When a load is applied to a specimen, it will drop further if the shearing stresses across the grain boundaries are relaxed. This further dropping of the load implies an increase in strain energy, and hence in the strain energy stored by each grain. It may not be obvious, at first sight, how such a relaxation of shearing stresses at the boundaries of a grain increases its total strain energy. It does become obvious, however, when we recall a well-known mathematical theorem. This theorem states that that function φ which makes $(\varphi^2)_{\text{Av}}$ a minimum, subject to the condition that $(\varphi)_{\text{Av}}$ be a constant, is itself a constant. Any allowed deviation of φ from a constant function will necessarily raise the average of its square. Now the strain energy of a grain is an average of a quadratic function of the stresses. The average of the stresses themselves, however, is determined by the conditions of loading. Thus if the z axis is chosen as the axis of loading, then the average of all stresses is zero except $(Z_z)_{\text{Av}}$. If the grains are elastically isotropic, the stresses are everywhere constant before the grain boundaries slip. Slipping at the boundaries sets up an inhomogeneous distribution of stresses, and hence necessarily raises the strain energy. The same qualitative result is to be expected for elastically anisotropic grains. This complicating factor of elastic anisotropy will be neglected in the following discussion.

These qualitative considerations suggest that we attack our problem by comparing the strain energy stored in an individual grain for the two cases: (1) uniform stresses, (2) no shearing stresses across grain boundaries. The average of the stresses is to be the same in each case, namely, all are to vanish except $(Z_z)_{\text{Av}}$. In each case the modulus E may be obtained from the equation

$$W=(1/2E)(Z_z)_{\text{Av}}^2, \tag{1}$$

where W is the mean strain energy per unit volume. It is not to be expected that the ratio of the modulus E for the two cases will depend much upon the shape of the grain, as long as the grain has a high degree of symmetry. In particular, it is to be expected that the ratio will be practically the same for grains which are spheres as for grains which are pentagonal dodecahedrons. Since the theory of elasticity has been applied in detail to spheres, while not at all to dodecahedrons, we shall, in our calculations, assume the grains to be spherical.

§2. Solution of Problem

We shall first find that displacement vector $\mathbf{U}$ appropriate to our problem. This displacement vector must of course be a solution of the equations of equilibrium. It must also be associated with a zero shearing traction across the surface of a sphere of radius a. Upon taking the z axis to be our axis of loading, we have the further con-

[5] C. Desch, *The Chemistry of Solids* (Cornell, 1934), pp. 59–63.

dition that $\mathbf{U}$ must be such that the volume average of X_x and of Y_y vanish. This vector $\mathbf{U}$ we shall construct by superimposing three vector displacements $\mathbf{U}_{\mathrm{I}}$, $\mathbf{U}_{\mathrm{II}}$ and $\mathbf{U}_{\mathrm{III}}$, each of which are solutions of the equations of equilibrium. The first will be taken as the displacement vector corresponding to a uniform stress Z_z, all the other stresses vanishing.

$$\mathbf{U}_{\mathrm{I}} = e_1(-\sigma x,\ -\sigma y,\ z). \tag{2}$$

This represents the displacement vector in our polycrystal with axial loading, before relaxation of shearing stresses across grain boundaries.

Now $\mathbf{U}_{\mathrm{I}}$ is associated with the shearing stress

$$R_\theta = -e_1 E \cos\theta \sin\theta$$

across a spherical surface. The second displacement vector $\mathbf{U}_{\mathrm{II}}$ will be so chosen as to neutralize this shearing stress across the surface of a sphere of radius a. A solution of the equations of equilibrium which is associated with a shearing stress of this type is given by Love.[6] It is

$$\mathbf{U}_{\mathrm{III}} = e_2\{r^2(\partial/\partial x,\ \partial/\partial y,\ \partial/\partial z) + \alpha(x,\ y,\ z)\}(3z^2 - r^2)/a^2, \tag{3}$$

where α is given in terms of the Lamé elastic constants λ and μ by

$$\alpha = -2(2\lambda + 7\mu)/(5\lambda + 7\mu).$$

From the stresses, given by Love, which are associated with this displacement vector, we find that the shearing stress R_θ of $\mathbf{U}_{\mathrm{I}}$ is canceled if

$$e_2 = -e_1(E/12\mu)(5\lambda + 7\mu)/(8\lambda + 7\mu). \tag{4}$$

The third displacement vector $\mathbf{U}_{\mathrm{III}}$ will be so chosen as to neutralize the average of the transverse tensile stresses X_x and Y_y introduced by $\mathbf{U}_{\mathrm{II}}$. In order that no new shearing stresses be introduced across the spherical surface of radius a, $\mathbf{U}_{\mathrm{III}}$ must represent a pure dilatation,

$$\mathbf{U}_{\mathrm{III}} = e_3(x,\ y,\ z). \tag{5}$$

The averages of X_x and of Y_y due to $\mathbf{U}_{\mathrm{II}}$ are neutralized by taking

$$\begin{aligned} e_3 &= e_2(84\mu/15K)(\lambda + \mu)/(5\lambda + 7\mu) \\ &= -e_1(7E/15K)(\lambda + \mu)/(8\lambda + 7\mu). \end{aligned} \tag{6}$$

The displacement vector

$$\mathbf{U} = \mathbf{U}_{\mathrm{I}} + \mathbf{U}_{\mathrm{II}} + \mathbf{U}_{\mathrm{III}}, \tag{7}$$

with $\mathbf{U}_{\mathrm{I}}$, $\mathbf{U}_{\mathrm{II}}$, and $\mathbf{U}_{\mathrm{III}}$ given by Eqs. (2)–(6), now satisfies all conditions appropriate to our problem. From Eq. (1), the Young's modulus for the polycrystal with relaxed shearing stresses across grain boundaries is given by

$$E_R = (Z_z)_{\mathrm{Av}}{}^2/2W.$$

Here $(Z_z)_{\mathrm{Av}}$ and the mean strain energy density W are to be calculated with $\mathbf{U}$ of Eq. (7). In computing W, we need the total strain energy of the sphere. This may be most readily computed from the displacement $\mathbf{U}$ and the traction across the surface, $\mathbf{T}$,[7]

$$\text{Strain energy} = \tfrac{1}{2}\int \mathbf{U}\cdot\mathbf{T}\,dS,$$

the integral being taken over the surface of the sphere. Since $\mathbf{T}$ is normal to the surface of the sphere, the integrand reduces to $U_r T_r$. A straightforward but tedious calculation gives

$$E_R = \tfrac{1}{2}\{(7 + 5\sigma)/(7 + \sigma - 5\sigma^2)\}E, \tag{8}$$

where σ is Poisson's ratio. A plot of E_R/E is given in Fig. 1.

[6] A. E. H. Love, *Mathematical Theory of Elasticity* (Cambridge, 1934), 4th edition, pp. 250–251.

[7] R. V. Southwell, *Theory of Elasticity* (Oxford, 1936), pp. 8–9.